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ON PSEUDO-RESOLVENTS OF LINEAR INTEGRAL EQUATIONS IN GENERAL ANALYSIS.

BY T. H. HILDEBRANDT.

The purpose of this note is to extend the theory of pseudo-resolvents in linear integral equations as given by W. A. Hurwitz* to the General Analysis situation. This extension is a perfectly obvious one, if the basis is that used by Moore† for the Hilbert-Schmidt theory. For in this theory it is possible to carry through the argument exactly as given by Hurwitz. However, the Hurwitz theory is intended to replace the theory based on Fredholm minors, and so it would be desirable to operate on the basis which Moore uses for the Fredholm theory. We show that this is possible if we add an additional postulate on the J-operation in the case of one theorem: Theorem 5.

We assume that we are working in a basis (Σ_6):‡

$$\left(\mathfrak{A};\ \begin{matrix}\tilde{\mathfrak{P}},\ \tilde{\mathfrak{M}},\ \mathfrak{K} = (\tilde{\mathfrak{M}}\hat{\mathfrak{M}})_* \\ \hat{\mathfrak{P}},\ \hat{\mathfrak{M}},\ \check{\mathfrak{K}} = (\hat{\mathfrak{M}}\tilde{\mathfrak{M}})_*\end{matrix} \qquad J \text{ on } \check{\mathfrak{K}} \text{ to } \mathfrak{A}\right)$$

where $\mathfrak{A}$ is the class of complex numbers, $\tilde{\mathfrak{P}}$ and $\hat{\mathfrak{P}}$ are general classes of elements: $\tilde{p}$ and $\hat{p}$; $\tilde{\mathfrak{M}}$ and $\hat{\mathfrak{M}}$ are classes of functions ($\tilde{\mu}$, $\tilde{\xi}$, $\tilde{\eta}$; $\hat{\mu}$, $\hat{\xi}$, $\hat{\eta}$) on $\tilde{\mathfrak{P}}$, $\hat{\mathfrak{P}}$ respectively to $\mathfrak{A}$, and have the properties L, C and D; the operator J has the properties L and M, where in accordance with a recent simplification of Professor Moore's, *J has the property M*, if for every ν the values of $|J\nu_1|$ for the functions $|\nu_1| \leqq |\nu|$ have an upper bound. If $M\nu$ is defined to be the least upper bound of such values then M is a functional operation on $\mathfrak{N}$ to the non-negative real part of $\mathfrak{A}$, of such a nature that

$$\text{if } |\nu_1| \leqq |\nu| \text{ then } |J\nu_1| \leqq M\nu \text{ and } M\nu_1 \leqq M\nu.$$

We shall assume the following results derivable from the existence of the Fredholm determinant and first minor in this basis:

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[*] Cf. Transactions of the American Mathematical Society, vol. 13 (1912), pp. 405–418; 16 (1915), pp. 121–133. The desirability of such an extension has recently been mentioned by Evans: Cambridge Colloquium, p. 115. The present note gives a general case including the two papers of Hurwitz as special cases.

[†] Cf. Bulletin of the American Mathematical Society, ser. 2, vol. 18 (1912), p. 361, and Proceedings of Fifth International Congress, Cambridge, vol. I, p. 232.

[‡] Cf. Moore, loc. cit., p. 352.

(*A*) If the equation $\tilde{\xi} + J\kappa\tilde{\xi} = 0$ has a solution $\tilde{\xi}$ which is not identically zero, then the adjoint equation $\hat{\xi} + J\hat{\xi}\kappa = 0$ has a solution $\hat{\xi}$ not identically zero, and conversely.

(*B*) The Fredholm determinant $F_0(\kappa; z)$ of $z\kappa$ is an entire function of z. If κ_1 and κ_2 are such that $J\kappa_1\kappa_2 = 0$, then

$$F_0(\kappa_1 + \kappa_2; z) = F_0(\kappa_1; z)F_0(\kappa_2; z).$$

We also note the following well-known result from the theory of linear independence of functions:

(*C*) A necessary and sufficient condition that n functions $\mu_1, \cdots, \mu_n$ on $\mathfrak{P}$ to $\mathfrak{A}$ be linearly independent, is that there exist values $p_1, \cdots, p_n$ of the range $\mathfrak{P}$ such that the determinant of the elements $\mu_i(p_j)$ is not zero. As a consequence, if $\mu_1, \cdots, \mu_n$ are linearly independent, it is possible to find linear combinations μ_{0i} of the μ_i such that $\mu_{0i}(p_j) = \delta_{ij}$ (where δ_{ij} is the Kronecker δ, i.e., zero for $i \neq j$ and unity for $i = j$) and such that the set μ_{0i} is linearly equivalent to the set μ_i.

We then have the following theorems:

1. *The number of linearly independent solutions of* $\tilde{\xi} + J\kappa\tilde{\xi} = 0$ *is finite.*

Let $\tilde{\mu}_1, \cdots, \tilde{\mu}_n$ be a set of linearly independent solutions of $\tilde{\xi} + J\kappa\tilde{\xi} = 0$, which according to (*C*) can be chosen in such a way that there exist $\tilde{p}_j$ for which $\tilde{\mu}_i(\tilde{p}_j) = \delta_{ij}$. Then if $\kappa(\tilde{p}_j) = \hat{\mu}_j$ (which is obviously a function of the class $\widehat{\mathfrak{M}}$), we have:

$$J\hat{\mu}_i\tilde{\mu}_j = J\kappa(\tilde{p}_i)\tilde{\mu}_j = -\tilde{\mu}_i(\tilde{p}_j) = -\delta_{ij}.$$

Let

$$\kappa_1 = \kappa - \Sigma_1{}^n\tilde{\mu}_i\hat{\mu}_i \qquad \text{and} \qquad \kappa_2 = \Sigma_1{}^n\tilde{\mu}_i\hat{\mu}_i.$$

Then

$$J\kappa_1\kappa_2 = J(\kappa - \Sigma\tilde{\mu}_i\hat{\mu}_i)\Sigma\tilde{\mu}_j\hat{\mu}_j = 0.$$

Hence

$$\begin{aligned} F_0(\kappa; z) &= F_0(\kappa - \Sigma\tilde{\mu}_i\hat{\mu}_i + \Sigma\tilde{\mu}_i\hat{\mu}_i; z) \\ &= F_0(\Sigma\tilde{\mu}_i\hat{\mu}_i; z)F_0(\kappa - \Sigma\tilde{\mu}_i\hat{\mu}_i; z) = (1 - z)^n F_0(\kappa - \Sigma\tilde{\mu}_i\hat{\mu}_i; z). \end{aligned}$$

Since $F_0(\kappa; z)$ is an entire function of z, it can have the root $z = 1$ at most a finite number of times. Hence n is bounded.

A similar result holds for the equation $\hat{\xi} + J\hat{\xi}\kappa = 0$.

2. *The equation* $\tilde{\xi} + J\kappa\tilde{\xi} = 0$ *and its adjoint* $\hat{\xi} + J\hat{\xi}\kappa = 0$ *have the same number of linearly independent solutions.*

Let $\tilde{\mu}_{0i}$ $(i = 1, \cdots, n)$ be a complete set of linearly independent solutions of $\tilde{\xi} + J\kappa\tilde{\xi} = 0$ of such a nature that $\tilde{\mu}_{0i}(\tilde{p}_j) = -\delta_{ij}$, and let $\kappa(\tilde{p}_j) = \hat{\mu}_j$, so that $J\hat{\mu}_i\tilde{\mu}_{0j} = \delta_{ij}$. Similarly let $\hat{\mu}_{0i}$ $(i = 1, \cdots, m)$ be a complete set of linearly independent solutions of $\hat{\xi} + J\hat{\xi}\kappa = 0$, such that

$\hat{\mu}_{0i}(\tilde{p}_j) = -\delta_{ij}$ and let $\kappa(\hat{p}_j) = \tilde{\mu}_j$ so that $J\hat{\mu}_{0i}\tilde{\mu}_j = \delta_{ij}$. Suppose $m < n$. Then set $\kappa_0 = \kappa + \Sigma_1{}^m\tilde{\mu}_i\hat{\mu}_i$. Then by following out the method used by Hurwitz, it can be shown that the equation $\hat{\xi} + J\hat{\xi}\kappa_0 = 0$ has no non-identically zero solution, while $\tilde{\xi} + J\kappa_0\tilde{\xi} = 0$ has at least the solution μ_{0m+1}, which by the result (A) is impossible.

As an immediate consequence it follows that κ_0 has a reciprocal λ. If we replace κ_0 by its equivalent in the reciprocal relations which it satisfies, and make a slight transformation, we find that

3. $\kappa + \lambda + J\kappa\lambda + \Sigma_1{}^n\tilde{\mu}_i\hat{\mu}_{0i} = 0,$
   $\kappa + \lambda + J\lambda\kappa + \Sigma_1{}^n\tilde{\mu}_{0i}\hat{\mu}_i = 0.$

By using these relations, we prove at once:

4. *A necessary and sufficient condition that* $\tilde{\xi} + J\kappa\tilde{\xi} = \tilde{\eta}$ *shall have a solution* $\tilde{\xi}$ *is that* $J\hat{\mu}_{0i}\tilde{\eta} = 0$ *for every* i. *The solution is expressible in the form*

$$\tilde{\xi} = \tilde{\eta} + J\lambda\tilde{\eta} + \Sigma c_i\tilde{\mu}_{0i},$$

where the c_i *are arbitrary constants.*

Hurwitz's definition of pseudo-resolvents in this situation reads:

λ *is a pseudo-resolvent of* κ if the solution of the equation

$$\tilde{\xi} + J\kappa\tilde{\xi} = \tilde{\eta}$$

in which $J\hat{\mu}_{0i}\tilde{\eta} = 0$ for $i = 1, \cdots, n$ ($\hat{\mu}_{0i}$ being any complete set of linearly independent solutions of the adjoint homogeneous equation $\hat{\xi} + J\hat{\xi}\kappa = 0$) is expressible in the form

$$\tilde{\xi} = \tilde{\eta} + J\lambda\tilde{\eta} + \Sigma c_i\tilde{\mu}_{0i}$$

where $\tilde{\mu}_{0i}$ constitute a complete set of solutions of the homogeneous equation $\tilde{\xi} + J\kappa\tilde{\xi} = 0$.

For the following theorem it is desirable to introduce another postulate on the J-operation,[*] which is a generalization of the definite property of the integral. It is as follows:

If $J\hat{\mu}_0\tilde{\mu} = 0$ for every $\tilde{\mu}$ of the class $\tilde{\mathfrak{M}}$, then $\hat{\mu}_0 = 0$.

This property is equivalent to:

If $J\kappa_0\tilde{\mu} = 0$ for every $\tilde{\mu}$ of the class $\tilde{\mathfrak{M}}$, then $\kappa_0 = 0$. If J has this property, we shall say that it is *definite on the left.* Similarly J is *definite on the right,* if $J\hat{\mu}\tilde{\mu}_0 = 0$ or $J\hat{\mu}\kappa_0 = 0$ for every $\hat{\mu}$ of the class $\hat{\mathfrak{M}}$ has as consequence $\tilde{\mu}_0 = 0$ and $\kappa_0 = 0$, respectively. If J is definite on the right and left, then it is said to be *definite.*

We then have the theorem:

5. *If* J *is definite on the left, a necessary and sufficient condition that* λ

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[*] Cf. Transactions of the American Mathematical Society, vol. 19 (1918), p. 101.

be a pseudo-resolvent of κ *is that*

$$\kappa + \lambda + J\kappa\lambda + \Sigma_1{}^n\tilde{\mu}_i\hat{\mu}_{0i} = 0,$$

$$\kappa + \lambda + J\lambda\kappa + \Sigma_1{}^n\tilde{\mu}_{0i}\hat{\mu}_i = 0,$$

where the $\tilde{\mu}_{0i}$ *and* $\hat{\mu}_{0i}$ *constitute complete sets of linearly independent solutions of the homogeneous equations* $\tilde{\xi} + J\kappa\tilde{\xi} = 0$ *and* $\hat{\xi} + J\hat{\xi}\kappa = 0$, *respectively, and* $J\hat{\mu}_i\tilde{\mu}_{0j} = J\hat{\mu}_{0i}\tilde{\mu}_j = \delta_{ij}$.

The sufficiency of this condition without the additional postulate on the J is shown as in Theorem 4.

In order to show the necessity, we make a study of the function:

$$\kappa_1 = \kappa + \lambda + J\kappa\lambda.$$

It has the following properties:

(*a*) $\hat{\mu}_{0i} + J\hat{\mu}_{0i}\kappa_1 = 0$ for every i.
For

$$J\hat{\mu}_{0i}\kappa_1 = J\hat{\mu}_{0i}\kappa + J\hat{\mu}_{0i}\lambda + JJ\hat{\mu}_{0i}\kappa\lambda = J\hat{\mu}_{0i}\kappa = -\hat{\mu}_{0i}.$$

As a consequence the equation $\hat{\xi} + J\hat{\xi}\kappa_1 = 0$ has at least the n linearly independent solutions $\hat{\mu}_{01}, \cdots, \hat{\mu}_{0n}$.

(*b*) $J\hat{\mu}_{0i}(\kappa - \kappa_1) = 0$ for every i.
This is a direct consequence of (*a*).

(*c*) If $\tilde{\mu}$ is such that $J\hat{\mu}_{0i}\tilde{\mu} = 0$ for every i, then $J\kappa_1\tilde{\mu} = 0$ and conversely. For suppose $J\hat{\mu}_{0i}\tilde{\mu} = 0$ for every i. Then the equation $\tilde{\mu} = \tilde{\xi} + J\kappa\tilde{\xi}$ has a solution:

$$\tilde{\xi} = \tilde{\mu} + J\lambda\tilde{\mu} + \Sigma c_i\tilde{\mu}_{0i}.$$

If we substitute this value of ξ in the equation $\tilde{\mu} = \tilde{\xi} + J\kappa\tilde{\xi}$, we find that $J\kappa_1\tilde{\mu} = 0$. The converse is obtained from (*a*) by multiplying by $\tilde{\mu}$ and operating with J.

(*d*) $J\kappa_1(\kappa - \kappa_1) = 0$.
This is an immediate consequence of (*b*) and (*c*).

(*e*) The function $J\kappa_1\kappa_1$ satisfies the condition:

$$J\kappa_1\kappa_1 + J(J\kappa_1\kappa_1)\kappa = 0.$$

For consider the expression $\kappa_1 + J\kappa_1\kappa$. We have:

$$J\hat{\mu}_{0i}(\kappa_1 + J\kappa_1\kappa) = J\hat{\mu}_{0i}\kappa_1 + JJ\hat{\mu}_{0i}\kappa_1\kappa = -\hat{\mu}_{0i} + \hat{\mu}_{0i} = 0.$$

Applying the result (*c*) we find:

$$J\kappa_1(\kappa_1 + J\kappa_1\kappa) = 0 \qquad \text{or} \qquad J\kappa_1\kappa_1 + J(J\kappa_1\kappa_1)\kappa = 0.$$

As a consequence

(*f*) $J\kappa_1\kappa_1 = \Sigma_1{}^n\bar{\mu}_i\hat{\mu}_{0i}$, and as a matter of fact $J\hat{\mu}_{0i}\bar{\mu}_j = \delta_{ij}$. For

$$JJ\hat{\mu}_{0j}\kappa_1\kappa_1 = \Sigma J\hat{\mu}_{0j}\bar{\mu}_i\hat{\mu}_{0i}.$$

But $JJ\hat{\mu}_{0j}\kappa_1\kappa_1 = \hat{\mu}_{0j}$, and the $\hat{\mu}_{0j}$ are linearly independent. Hence

$$J\bar{\mu}_{0i}\bar{\mu}_j = \delta_{ij}.$$

(*g*) The functions $\bar{\mu}_i$ satisfy the homogeneous equation $\bar{\xi} + J\kappa_1\bar{\xi} = 0$. For

$$JJ\kappa_1\kappa_1\kappa_1 = \Sigma J\kappa_1\bar{\mu}_i\hat{\mu}_{0i} = \Sigma\bar{\mu}_iJ\hat{\mu}_{0i}\kappa_1 = -\Sigma\bar{\mu}_i\hat{\mu}_{0i},$$

according as we multiply the expression $J\kappa_1\kappa_1$ by κ_1 on the left or right and operate with J. If we multiply the equality:

$$\Sigma J\kappa_1\bar{\mu}_i\hat{\mu}_{0i} = -\Sigma\bar{\mu}_i\hat{\mu}_{0i}$$

by $\bar{\mu}_j$ on the right and operate with J, we obtain:

$$J\kappa_1\bar{\mu}_j = -\bar{\mu}_j.$$

(*h*) Every solution of $\bar{\xi} + J\kappa_1\xi = 0$ is a linear combination of the $\bar{\mu}_i$ and every solution of $\hat{\xi} + J\hat{\xi}\kappa_1 = 0$ is a linear combination of the $\hat{\mu}_{0i}$.

For every solution of $\bar{\xi} + J\kappa_1\bar{\xi} = 0$ is also a solution of

$$\bar{\xi} - J(J\kappa_1\kappa_1)\bar{\xi} = 0,$$

and every solution of $\hat{\xi} + J\hat{\xi}\kappa_1 = 0$ is also a solution of $\hat{\xi} - J\hat{\xi}(J\kappa_1\kappa_1) = 0$. If we replace $J\kappa_1\kappa_1$ by its value $\Sigma_1{}^n\bar{\mu}_i\hat{\mu}_{0i}$, the result is immediate. It follows from this result that the $\bar{\mu}_i$ and $\hat{\mu}_{0i}$ constitute complete sets of solutions of the homogeneous equations in κ_1.

(*i*) $J(\kappa_1 + \Sigma\bar{\mu}_i\hat{\mu}_{0i})\bar{\mu} = 0$ for every $\bar{\mu}$ of the class $\mathfrak{M}$.

For every $\bar{\mu}$ is expressible in the form:

$$\bar{\mu} = \Sigma(J\hat{\mu}_{0i}\bar{\mu})\bar{\mu}_i + \bar{\mu}_0, \text{ with } J\hat{\mu}_{0i}\bar{\mu}_0 = 0 \text{ for every } i.$$

Now

$$J(\kappa_1 + \Sigma\bar{\mu}_i\hat{\mu}_{0i})(\Sigma c_j\bar{\mu}_j) = \Sigma_jc_jJ\kappa_1\bar{\mu}_j + \Sigma_{ij}c_j\bar{\mu}_iJ\hat{\mu}_{0i}\bar{\mu}_j$$

$$= -\Sigma_jc_j\bar{\mu}_j + \Sigma_jc_j\bar{\mu}_j = 0.$$

Obviously

$$J\bar{\mu}_i\hat{\mu}_{0i}\bar{\mu}_0 = \bar{\mu}_iJ\hat{\mu}_{0i}\bar{\mu}_0 = 0 \text{ for every } i$$

and by (*c*)

$$J\kappa_1\bar{\mu}_0 = 0.$$

Hence

$$J(\kappa_1 + \Sigma\bar{\mu}_i\hat{\mu}_{0i})\bar{\mu} = J(\kappa_1 + \Sigma_i\bar{\mu}_i\hat{\mu}_{0i})(\Sigma_jc_j\bar{\mu}_j + \bar{\mu}_0) = 0.$$

If J is definite on the left it follows that

$$\kappa_1 + \Sigma\bar{\mu}_i\hat{\mu}_{0i} = 0 \qquad \text{or} \qquad \kappa_1 = -\Sigma\bar{\mu}_i\hat{\mu}_{0i}.$$

In other words we have:

(j) $\kappa + \lambda + J\kappa\lambda + \Sigma\bar{\mu}_i\hat{\mu}_{0i} = 0$ with $J\hat{\mu}_{0i}\bar{\mu}_j = \delta_{ij}$.

In order to show the correctness of the second relation:

$$\kappa + \lambda + J\lambda\kappa + \Sigma\bar{\mu}_{0i}\hat{\mu}_i = 0$$

we notice that the equation

$$\lambda + J\kappa\lambda = -\kappa - \Sigma\bar{\mu}_i\hat{\mu}_{0i} = \kappa_1 - \kappa$$

fulfills the conditions required by the hypothesis of the theorem to enable us to solve for λ. We find:

$$\begin{aligned} \lambda &= -\kappa - \Sigma\bar{\mu}_i\hat{\mu}_{0i} - J\lambda(\kappa + \Sigma\bar{\mu}_i\hat{\mu}_{0i}) + \Sigma\bar{\mu}_{0i}\hat{\mu}_{1i} \\ &= -\kappa - J\lambda\kappa - \Sigma(\bar{\mu}_i + J\lambda\bar{\mu}_i)\hat{\mu}_{0i} + \Sigma\bar{\mu}_{0i}\hat{\mu}_{1i}. \end{aligned}$$

By multiplying the first relation by $\bar{\mu}_i$ and operating with J, we find:

$$J\kappa\bar{\mu}_i + J\lambda\bar{\mu}_i + JJ\kappa\lambda\bar{\mu}_i + \bar{\mu}_i = 0,$$

i.e.,

$$\bar{\mu}_i + J\lambda\bar{\mu}_i + J\kappa(\bar{\mu}_i + J\lambda\bar{\mu}_i) = 0$$

and so

$$\bar{\mu}_i + J\lambda\bar{\mu}_i = \Sigma_j c_{ij}\bar{\mu}_{0j}.$$

Hence if we let

$$\hat{\mu}_i = \hat{\mu}_{1i} + \Sigma_j\hat{\mu}_{0j}c_{ji},$$

$$\lambda + \kappa + J\lambda\kappa + \Sigma\bar{\mu}_{0i}\hat{\mu}_i = 0.$$

If we multiply this relation through by $\bar{\mu}_{0j}$ and operate with J, we find:

$$J\lambda\bar{\mu}_{0j} + J\kappa\bar{\mu}_{0j} + JJ\lambda\kappa\bar{\mu}_{0j} = -\Sigma_i\bar{\mu}_{0i}J\hat{\mu}_i\bar{\mu}_{0j}$$

or

$$\bar{\mu}_{0j} = \Sigma_i\bar{\mu}_{0i}J\hat{\mu}_i\bar{\mu}_{0j},$$

and hence on account of the linear independence of the $\bar{\mu}_{0j}$,

$$J\mu_i\bar{\mu}_{0j} = \delta_{ij}.$$

This completes the proof of the theorem.

If we define pseudo-resolvents in terms of the solution of the adjoint equation $\hat{\xi} + J\hat{\xi}\kappa = \hat{\eta}$, we get the same result as in Theorem 5, if we assume J to be definite on the right.

Obviously we can show that a pseudo-resolvent λ for the equation $\bar{\xi} + J\kappa\bar{\xi} = \bar{\eta}$ will also be a pseudo-resolvent for the adjoint equation $\hat{\xi} + J\hat{\xi}\kappa = \hat{\eta}$ and conversely.

If we drop the condition that J is definite on the left in Theorem 5 it becomes:

A necessary and sufficient condition that λ be a pseudo-resolvent is that

$$\kappa + \lambda + J\kappa\lambda + \Sigma\bar{\mu}_i\hat{\mu}_{0i} = \kappa_0,$$

$$\kappa + \lambda + J\lambda\kappa + \Sigma\bar{\mu}_{0i}\hat{\mu}_i = \kappa_0 + J\lambda\kappa_0,$$

where $J\hat{\mu}_{0i}\breve{\mu}_j = J\hat{\mu}_i\breve{\mu}_{0j} = \delta_{ij}$, and κ_0 is such that $J\kappa_0\breve{\mu} = 0$ for every $\breve{\mu}$ of the class $\breve{\mathfrak{M}}$, and $J\hat{\mu}_{0i}\kappa_0 = 0$ for every i. Such a λ will not serve as a pseudo-resolvent for $\hat{\xi} + J\hat{\xi}\kappa = \hat{\eta}$ unless we have also $J\hat{\mu}\kappa_0 = 0$ for every $\hat{\mu}$ of $\hat{\mathfrak{M}}$. Then $J\lambda\kappa_0 = 0$ also.

The following theorem is of importance in obtaining functions λ satisfying the pseudo-reciprocal relations of Theorem 5:

6. *A necessary and sufficient condition that a kernel of the form:*

$$\kappa_1 = \kappa + \Sigma_{ij}a_{ij}\breve{\mu}_i\hat{\mu}_j,$$

where $J\acute{\mu}_{0i}\breve{\mu}_j = J\hat{\mu}_i\breve{\mu}_{0j} = \delta_{ij}$, *have no zero solutions, i.e., be such that* $\breve{\xi} + \breve{J}\kappa\xi_1 = 0$ *has as consequence* $\breve{\xi} = 0$, *is that the determinant of the elements* a_{ij} *be different from zero. The reciprocal* λ *of any such* κ_1 *will satisfy the relations of Theorem 5, i.e., be a pseudo-resolvent of* κ.

Suppose that $\breve{\xi}$ is a solution of

$$\breve{\xi} + J\kappa_1\breve{\xi} = \breve{\xi} + J(\kappa + \Sigma_{ij}a_{ij}\breve{\mu}_i\hat{\mu}_j)\breve{\xi} = 0.$$

If we multiply by $\hat{\mu}_{0i}$ and operate with J, we get:

$$\Sigma_j a_{ij}J\hat{\mu}_j\breve{\xi} = 0,$$

and so

$$\breve{\xi} + J\kappa\breve{\xi} = 0, \qquad \text{i.e.,} \qquad \breve{\xi} = \Sigma_k c_k\breve{\mu}_{0k}.$$

In order to determine the c_k we substitute in the relation $\Sigma_j a_{ij}J\hat{\mu}_j\breve{\xi} = 0$, and obtain:

$$\Sigma_j a_{ij}c_j = 0.$$

A necessary and sufficient condition that $c_j = 0$ be the only solution of this set of equations is that the determinant of the a_{ij} be different from zero.

The second part of the theorem follows as in Theorem 3.

Obviously the determination of λ satisfying the pseudo-reciprocal is not unique. We note that:

7. *If* λ_1 *and* λ_2 *both satisfy the equations of Theorem* 5 *for the same* κ, *then they differ by a function of the form* $\Sigma_{ij}b_{ij}\breve{\mu}_{0i}\hat{\mu}_{0j}$.

For the difference $\lambda_1 - \lambda_2$ of two such functions will satisfy the equations:

$$\lambda_1 - \lambda_2 + J\kappa(\lambda_1 - \lambda_2) = 0,$$

$$\lambda_1 - \lambda_2 + J(\lambda_1 - \lambda_2)\kappa = 0,$$

from which it follows that $\lambda_1 - \lambda_2$ is expressible in the form $\Sigma_{ij}b_{ij}\breve{\mu}_{0i}\hat{\mu}_{0j}$

In the above two theorems it was assumed that $\breve{\mu}_{0i}$ and $\hat{\mu}_{0i}$ constituted complete sets of solutions of the homogeneous equations $\breve{\xi} + J\kappa\breve{\xi} = 0$ and $\hat{\xi} + J\hat{\xi}\kappa = 0$, respectively. If we drop this condition, we get:

8. If κ and λ satisfy a pair of equations of the form of the pseudo-reciprocal relations of Theorem 5:

$$\kappa + \lambda + J\kappa\lambda + \Sigma\tilde{\mu}_i\hat{\mu}_{0i} = 0,$$

$$\kappa + \lambda + J\lambda\kappa + \Sigma\tilde{\mu}_{0i}\hat{\mu}_i = 0,$$

in which $J\hat{\mu}_i\tilde{\mu}_{0j} = J\hat{\mu}_{0i}\tilde{\mu}_j = \delta_{ij}$ $(i, j = 1, \cdots, n)$ so that each of the four sets of functions $\tilde{\mu}_i$, $\hat{\mu}_i$, $\tilde{\mu}_{0i}$, $\hat{\mu}_{0i}$ constitutes a linearly independent set, then the solutions of the equation $\tilde{\xi} + J\kappa\tilde{\xi} = 0$ are linear combinations of $\tilde{\mu}_{0i}$; of the equation $\hat{\xi} + J\hat{\xi}\kappa = 0$, are linear combinations of $\hat{\mu}_{0i}$; and obviously on account of the symmetry, the solutions of the corresponding equations in the λ will be linear combinations of the $\tilde{\mu}_i$ and $\hat{\mu}_i$, respectively.

For suppose $\tilde{\mu} + J\kappa\tilde{\mu} = 0$. Then if we multiply the second of the relations of the theorem by $\tilde{\mu}$ and operate with J we find:

$$J\kappa\tilde{\mu} + J\lambda\tilde{\mu} + JJ\lambda\kappa\tilde{\mu} + \Sigma\tilde{\mu}_{0i}J\hat{\mu}_i\tilde{\mu} = -\tilde{\mu} + \Sigma\tilde{\mu}_{0i}J\hat{\mu}_i\tilde{\mu} = 0$$

or

$$\tilde{\mu} = \Sigma(J\hat{\mu}_i\tilde{\mu})\tilde{\mu}_{0i}.$$

The other results are obtained in a similar way.

If we multiply the first of the two relations of this theorem by $\hat{\mu}_{0i}$ on the right and operate with J we get:

$$J\hat{\mu}_{0i}\kappa + J\hat{\mu}_{0i}\lambda + JJ\mu_{0i}\kappa\lambda + \hat{\mu}_{0i} = 0,$$

i.e., $\hat{\mu}_{0i} + J\hat{\mu}_{0i}\kappa$ is a solution of the equation $\hat{\xi} + J\hat{\xi}\lambda = 0$. Similar results are obtainable for the other three homogeneous equations.

9. *If* κ *and* λ *satisfy the relations of Theorem* 8, *and if the equation* $\tilde{\xi} + J\lambda\tilde{\xi} = 0$ *has no solution excepting* $\tilde{\xi} = 0$, *then the* $\tilde{\mu}_{0i}$ *and* $\hat{\mu}_{0i}$ *are the solutions of* $\tilde{\xi} + J\kappa\tilde{\xi} = 0$ *and* $\hat{\xi} + J\hat{\xi}\kappa = 0$ *respectively; and* λ *is a reciprocal of a function of the form*:

$$\kappa_1 = \kappa + \Sigma_{ij}a_{ij}\tilde{\mu}_i\hat{\mu}_j.$$

The first part of the theorem follows immediately from the remarks at the close of Theorem 8. As for the second part, suppose that κ_1 is a reciprocal of λ. Then

$$\kappa_1 + \lambda + J\kappa_1\lambda = 0.$$

Subtracting this from the first of the pseudo-reciprocal relations, we get:

$$\kappa - \kappa_1 + J(\kappa - \kappa_1)\lambda + \Sigma\tilde{\mu}_i\hat{\mu}_{0i}.$$

Since κ_1 is a reciprocal of λ we have:

$$\kappa - \kappa_1 = -\Sigma\tilde{\mu}_i(\hat{\mu}_{0i} + J\hat{\mu}_{0i}\kappa_1).$$

In a similar way we show that

$$\kappa - \kappa_1 = -\Sigma(\tilde{\mu}_{0i} + J\kappa_1\tilde{\mu}_{0i})\hat{\mu}_i,$$

and so

$$\kappa - \kappa_1 = -\Sigma_{ij}a_{ij}\tilde{\mu}_i\hat{\mu}_j.$$